

Strictly Isospectral Potentials From Excited Quantum States

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The strictly nonrelativistic isospectral scheme based on the general Riccati solution and Darboux transformation function corresponding to excited states is presented on the following examples: the harmonic oscillator, the square well, and the hydrogen atom.

Supersymmetric quantum mechanics (SUSY QM) (Witten, 1981; for a review see, e.g., Rosu, 1999) is based on Riccati equations of the form

$$\frac{dy}{dz} + y^2 = \tilde{V}_1(z) + \epsilon, \quad (1)$$

where $\tilde{V}_1(z)$ is the initial Schrödinger potential and $\epsilon < 0$ is the so-called factorization energy that for convenience we shall absorb in the potential, i.e., we shall redefine the initial potential as $V_1 = \tilde{V}_1 + \epsilon$. The SUSY partner Riccati equation will be

$$-\frac{dy}{dz} + y^2 = V_2(z), \quad (2)$$

where the partner potential $V_2(z)$ is Darboux “isospectral” with respect to $V_1(z)$, i.e.

$$V_2(z) = V_1(z) - 2D^2[\ln(\psi(z))], \quad (3)$$

where $D = \frac{d}{dz}$ and ψ is a particular solution of the Schrödinger equation $D^2\psi - V_1\psi = 0$. The only difference between the spectra of V_2 and V_1 is that the energy level corresponding to the wave function $\psi(z)$ used as transformation function is missing from the spectrum of V_2 .

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In 1984, Mielnik introduced a SUSY QM scheme based on the general Riccati solution with application to the harmonic oscillator case (Mielnik, 1984; for recent applications see Rosu, 1996, 1997; Rosu and Reyes, 1998). In the latter approach, one gets a one-parameter family of Darboux strictly isospectral potentials with respect to V_1 given by

$$V_1(z; \lambda) = V_1(z) - 2D^2[\ln(I(z) + \lambda)], \quad (4)$$

where $I(z) = \int_l^z \psi^2(x) dx$. Moreover,

$$\psi(z, \lambda) = \frac{\sqrt{\lambda(\lambda + 1)}\psi(z)}{\int_l^z \psi^2(x) dx + \lambda} \quad (5)$$

is the modulated Schrödinger mode implied by this scheme, in which the Riccati integration constant λ is kept as a free parameter. The inferior limit l of the integral $I(z)$ is zero for the radial problems, $-\infty$ for the full-line problems, and $-L/2$ for the square-well case (see below). The factor $\sqrt{\lambda(\lambda + 1)}$ is a normalization constant of the strictly isospectral SUSY modes, which interestingly does not depend on the quantum numbers. In general, to get continuous solutions $\psi(z, \lambda)$, one should avoid a zero denominator. This leads to conditions on the possible values of λ . On the other hand, if one works with polynomial solutions, which is the usual case for the discrete spectrum of an exactly solvable quantum problem, there will be singularities in the logarithmic derivative of ψ . SUSY partner potentials based on the n 'th excited state of a Schrödinger discrete spectrum problem split in $n + 1$ branches separated by the n singularities of the logarithmic derivative. Such problems have been considered by Robnik (1997) and Robnik and Liu (\dots) (see also Fernandez *et al.*, 1999).

In the following, we focus on Robnik's results from the perspective of the strictly isospectral scheme, namely instead of working with Eq. (3) as Robnik did, we make use of Eqs. (4) and (5). This is also equivalent to saying that we work with factorization energies $\epsilon = \epsilon_n$ in Eq. (1) and $\psi = \psi_n$ as transformation function in Eq. (4), where ϵ_n is an eigenvalue of the discrete spectrum of the exactly solvable problem and ψ_n is the corresponding eigenfunction. Our applications are the following:

(i) *The one-dimensional harmonic oscillator*

The initial potential is

$$V_1(z) = z^2 - (2n + 1) \quad (6)$$

for $n \in 0, 1, 2, \dots$. The wave functions are given in terms of the Hermite polynomials in the form

$$\psi_n = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(z) \exp(-z^2/2). \quad (7)$$

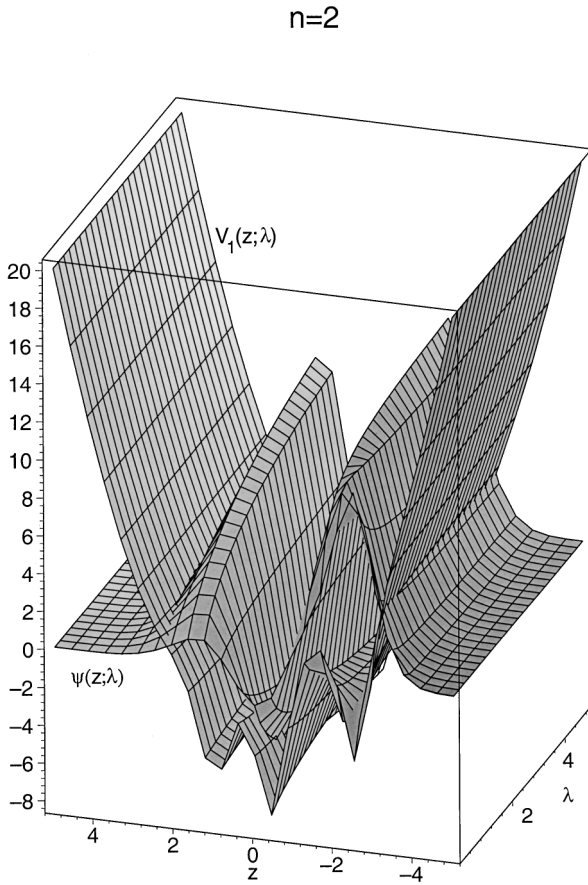


Fig. 1. The strictly isospectral potential and the deformed wave function (multiplied by 10 for the sake of visualization) obtained using the $n = 2$ excited state of the harmonic oscillator as Darboux transformation function.

We present plots of Eqs. (4) and (5) corresponding to $n = 2$ and $n = 3$ in Figs. 1 and 2, respectively. The examination of the plots shows that there are no singularities in the one-parameter family of potentials for the allowed range of the λ parameter. Instead, there are $n + 1$ wells, where n is the number of nodes of the excited state on which the strictly isospectral SUSY construction is based. Thus, we have at our disposal a definite method of producing multiple-well oscillator potentials of exactly solvable type.

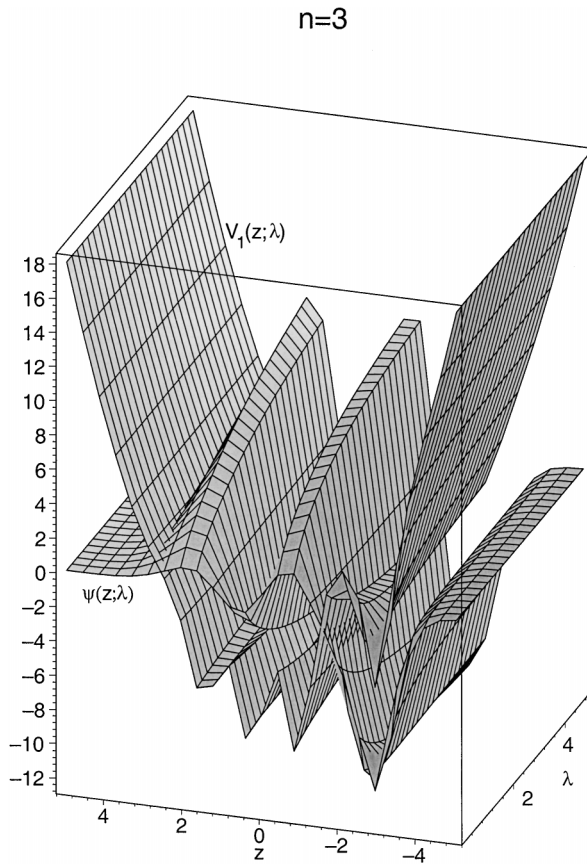


Fig. 2. The strictly isospectral potential and the deformed wave function (with the same scale factor of 10) obtained using the $n = 3$ excited state of the harmonic oscillator as Darboux transformation function.

(ii) *One-dimensional square well of length L*

The initial potential is

$$V_1(z) = \begin{cases} -\frac{\pi^2}{L^2}n^2 & \text{for } -L/2 \leq z \leq L/2, \\ V_1(z) = +\infty & \text{otherwise,} \end{cases} \quad (8)$$

where n belongs to the set $n = 1, 2, \dots$. The wave functions within the well are

$$\psi_n = \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi z}{L}\right) \quad \text{for odd } n, \quad (9)$$

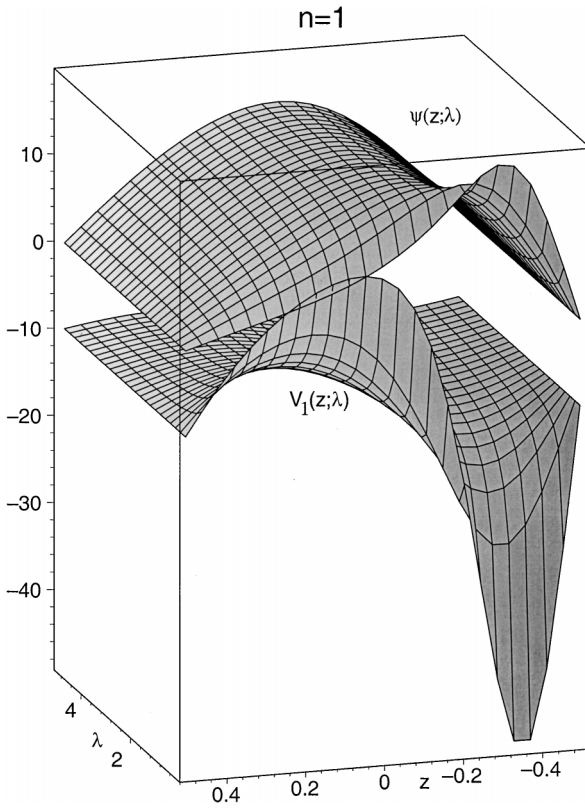


Fig. 3. The strictly isospectral potential and the deformed wave function ($\times 10$) obtained using the $n = 1$ excited state of the square well as Darboux transformation function.

and

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi z}{L}\right) \quad \text{for even } n. \tag{10}$$

Plots of the strictly isospectral formulas for this case are given in Figs. 3 and 4 for an odd case and an even case, respectively.

(iii) *The three-dimensional Kepler problem*

The initial potential is

$$V_1(z) = -\frac{2}{z} + \frac{l(l+1)}{z^2} - \frac{4}{(n_r + l + 1)^2} \tag{11}$$

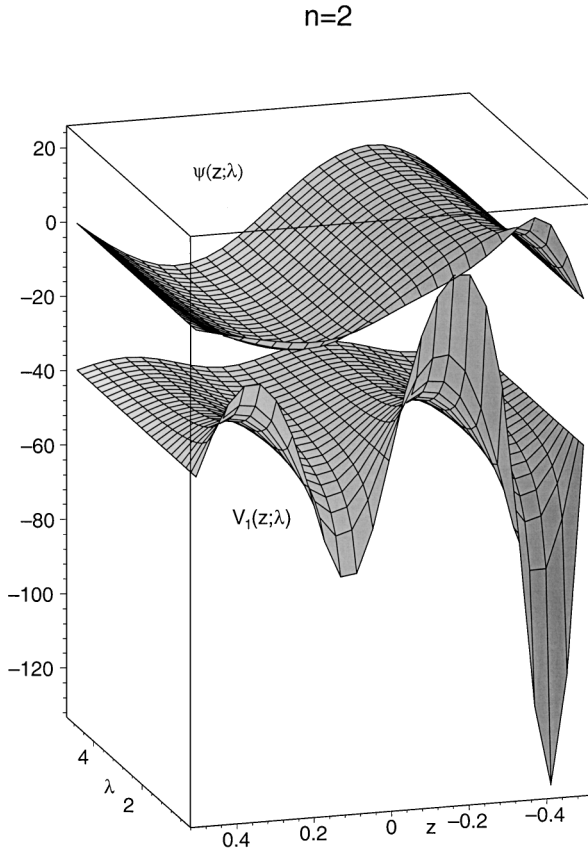


Fig. 4. The strictly isospectral potential and the deformed wave function ($\times 10$) obtained using the $n = 2$ excited state of the square well as Darboux transformation function.

for n_r from the set $n_r = 0, 1, 2, \dots$, and where the centrifugal barrier has also been included.

The radial wave functions to be used as Darboux transformation functions are

$$R_{nl} = -\frac{2}{n^2} \sqrt{\frac{(n-l-1)!}{2n[(n+l)!]^3}} z^l e^{-z/2} L_{n+l}^{2l+1}(z) \tag{12}$$

where $z = 2r/n$ and the Bohr radius has been chosen as unity.

In Figs. 5–7 we present the result of the strictly isospectral construction based on the hydrogen radial wave functions R_{30} and R_{21} ,

$n=3 \quad l=0$

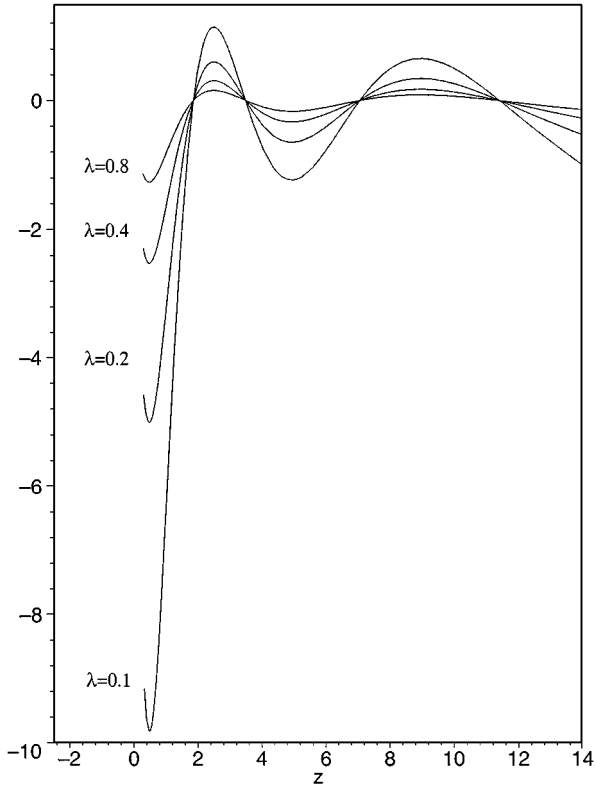


Fig. 5. The percent difference $100(V_1(z; \lambda) - V_1(z)) / V_1(z)$ between the strictly isospectral Coulomb potential and the Coulomb potential when the hydrogen radial state R_{30} is used as Darboux transformation function for four values of the parameter λ .

respectively. For the employed range of the λ parameter, there is a small strictly isospectral effect extending in the Rydberg region till about 20 Bohr radii. On the other hand, although not shown in the plots, we can report that in the $z = 0$ nuclear region the effect can be as big as 25%.

We recall that for this case a similar approach has been used by Fernández (1984). However, his procedure cannot be applied to the $l = 0$ cases, whereas the way we use the strictly isospectral Darboux scheme covers the full hydrogen spectrum.

$$n=3 \quad l=0$$

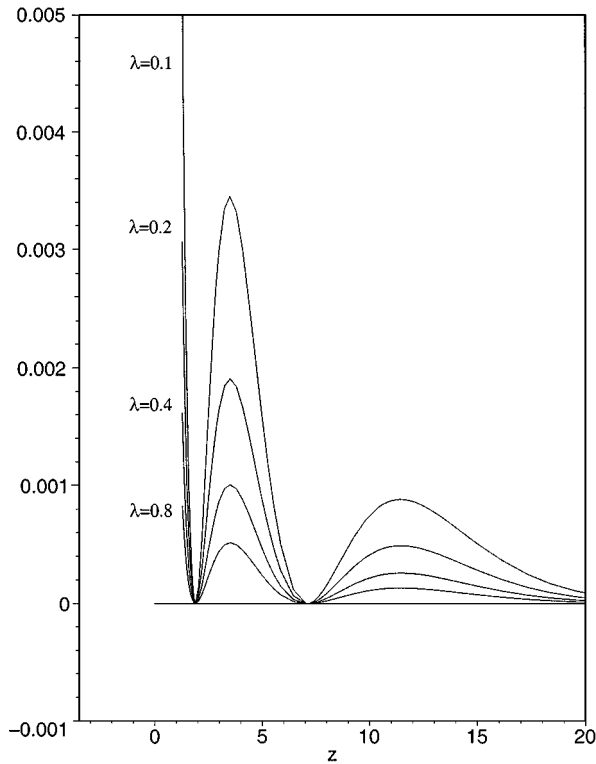


Fig. 6. The difference between the probability densities $\psi^2(z; \lambda) - \psi^2(z)$ for the same case as in Fig. 5.

Finally, we notice that it is still an open issue to what definite physical situation the strictly isospectral scheme does apply. There are hints in the literature (Boya *et al.*, 1998; Monthus *et al.*, 1996) indicating that λ can be used as a measure of confining effects on the quantum spectra.

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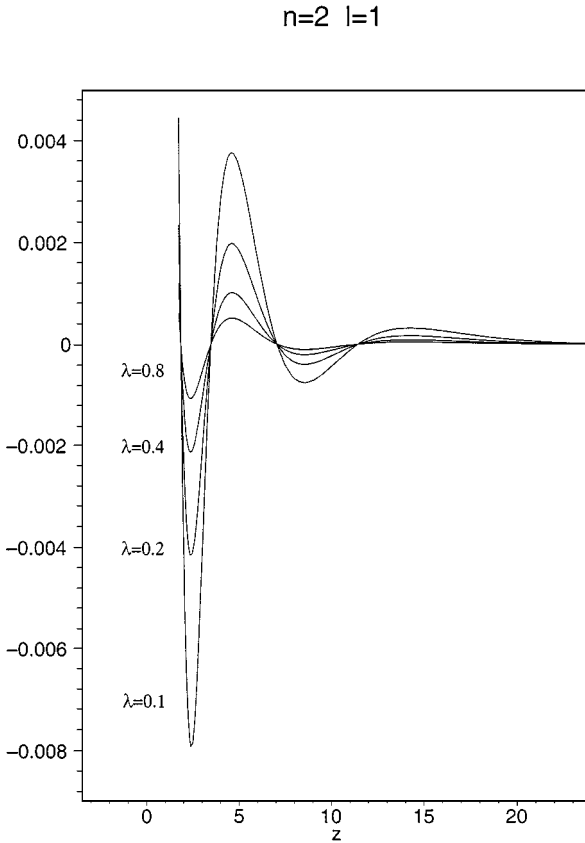


Fig. 7. The difference between the strictly isospectral Coulomb potential and the Coulomb potential $V_1(z; \lambda) - V_1(z)$ when the hydrogen radial state R_{21} is used as Darboux transformation function for the same four values of the parameter λ .

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